A Parameterized Algorithm for Upward Planarity Testing of Biconnected Graphs

Master's thesis presentation

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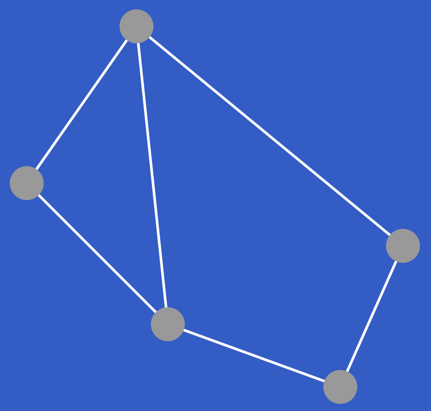
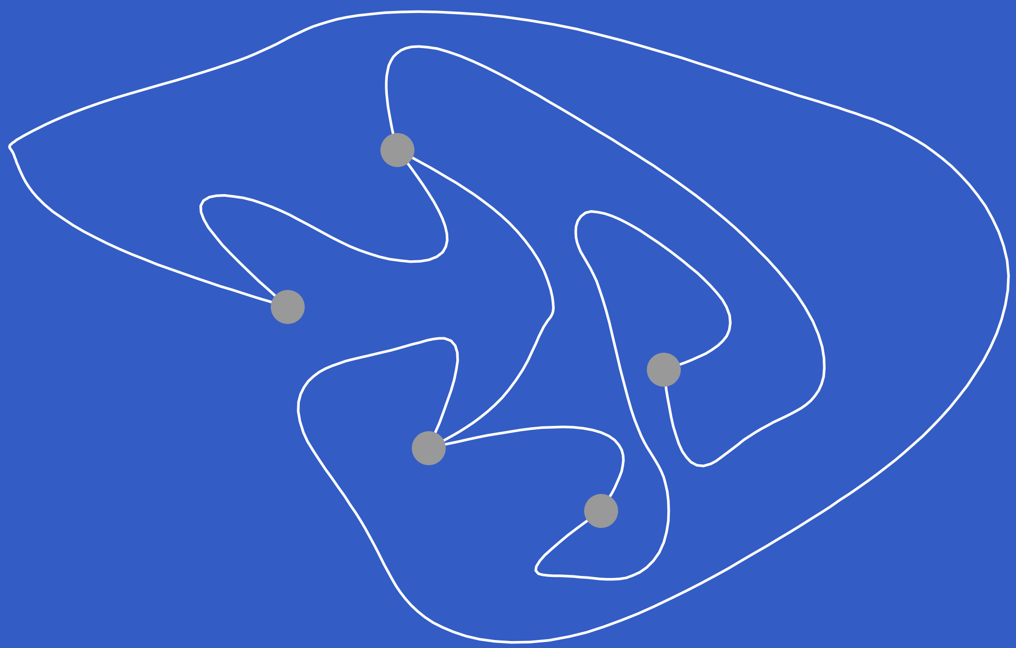
Graph drawing

- goal: visualization of graph structures
- vertices represented by points, edges by curves
- want drawings to satisfy certain criteria



Straight line drawing



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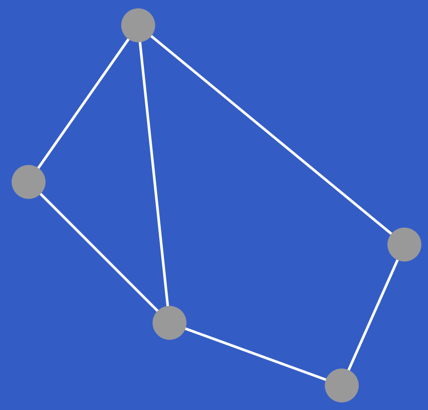
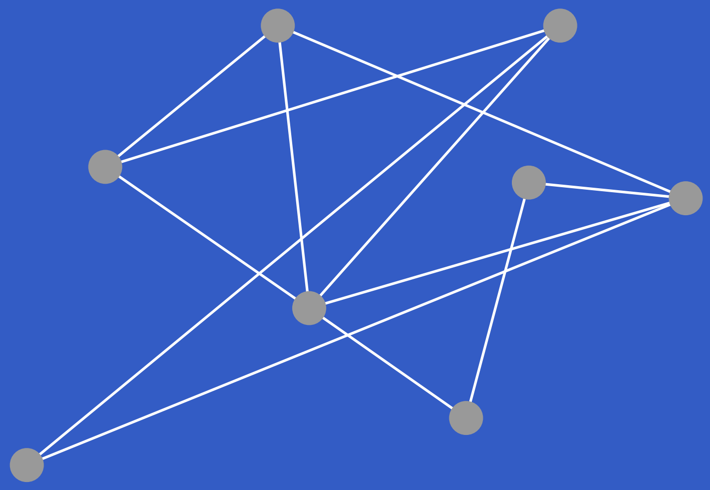


Planarity



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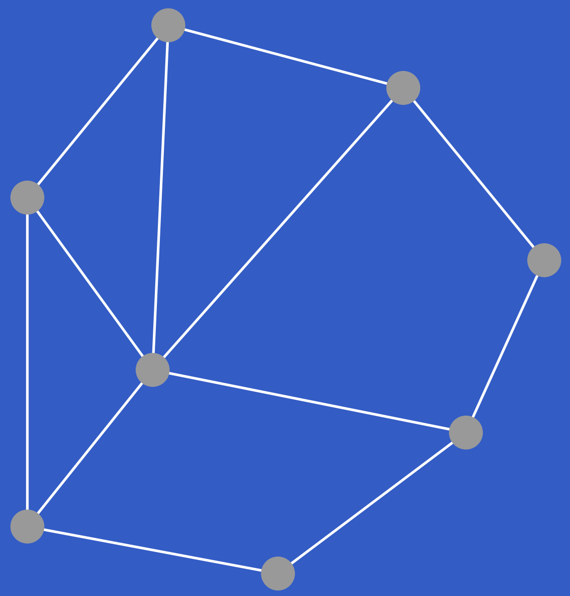

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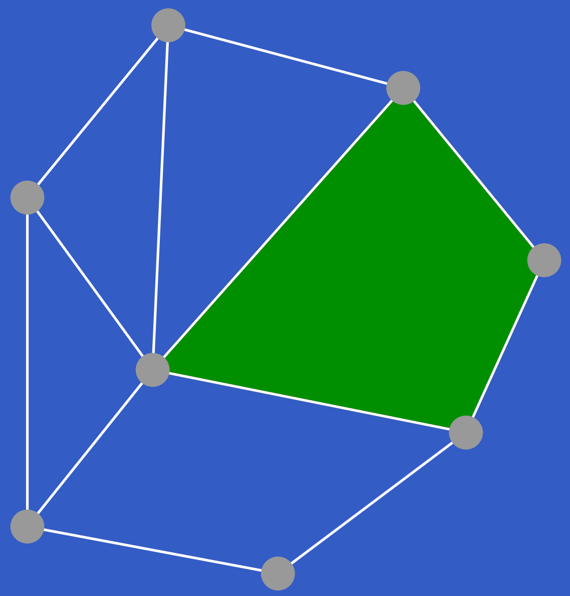
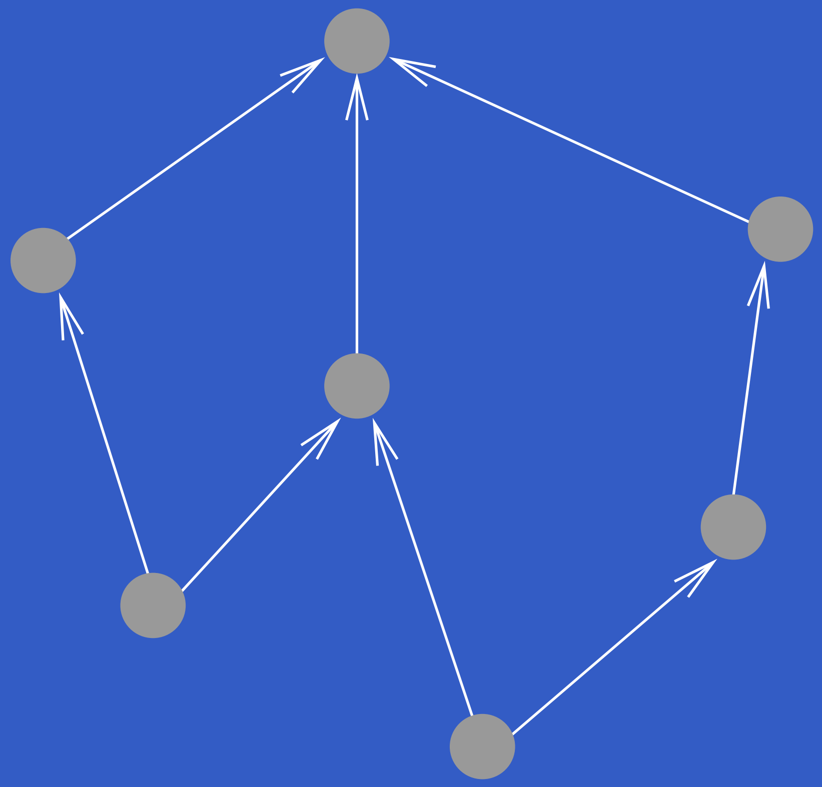
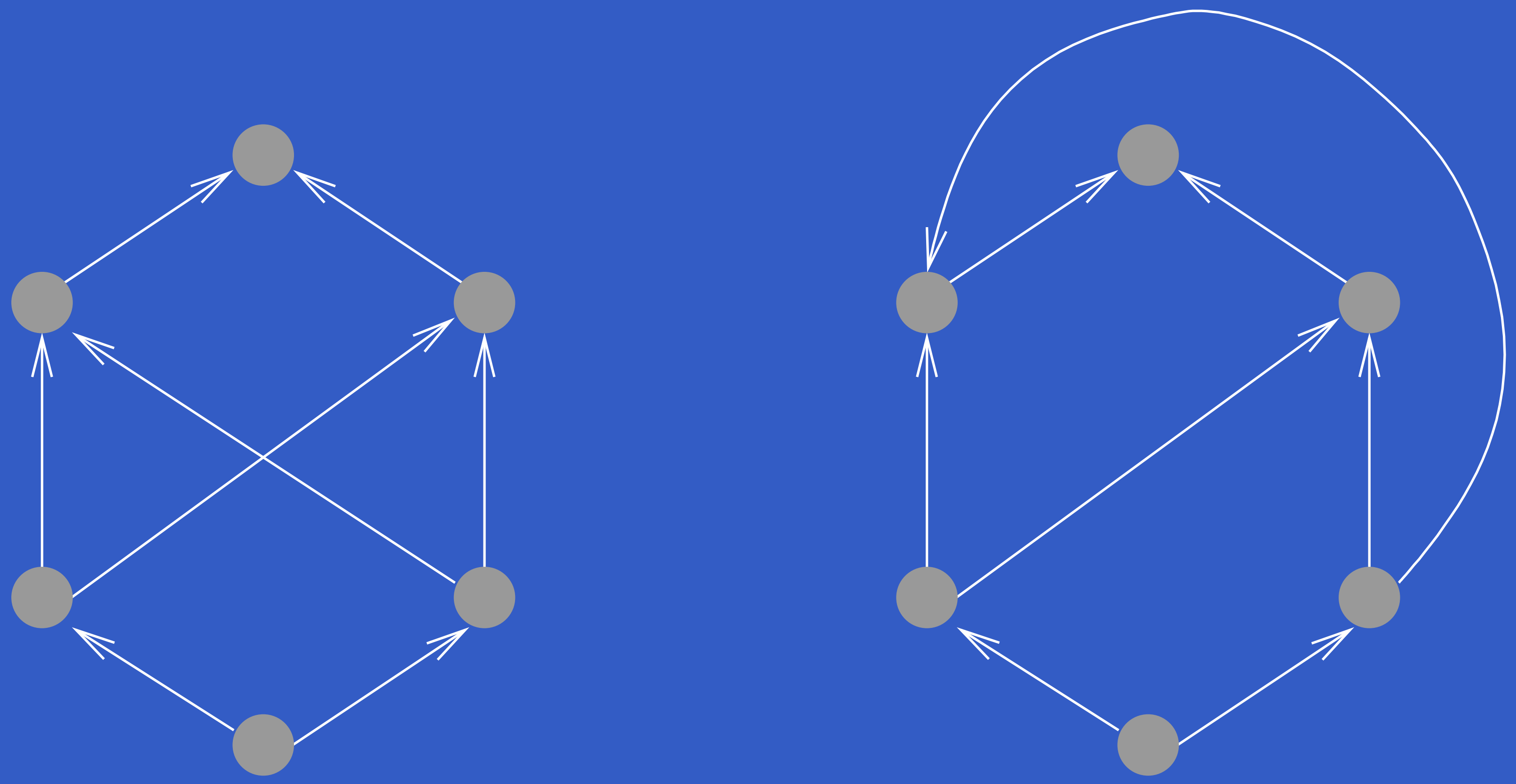


Upward planarity



Upward planarity



Our goal

- Find an efficient solution to upward planarity testing.
- Testing for planarity is linear time (Hopcroft and Tarjan 1974)
- Testing for upward (crossings allowed) is linear time (e.g. Cormen et al. 2001, Brassard and Bratley 1996)

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- Find an efficient solution to upward planarity testing.
- Testing for planarity is linear time (Hopcroft and Tarjan 1974)
- Testing for upward (crossings allowed) is linear time (e.g. Cormen et al. 2001, Brassard and Bratley 1996)
- Unfortunately, upward planarity testing is NP-complete (Garg and Tamassia 2001).

Related work

Class	**Complexity**	**Reference**
st-graph	$O(n)$	Di Battista and Tamassia 1988
bipartite	$O(n)$	Di Battista, Liu, and Rival 1990
triconnected	$O(n + r^2)$	Bertolazzi et al. 1994
outerplanar	$O(n^2)$	Papakostas 1995
single source	$O(n)$	Bertolazzi et al. 1998

Parameterized complexity

- developed by Downey and Fellows
- limit combinatorial explosion to some aspect of the problem

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  - NP-complete
  - to find a vertex cover of size k: $O\left(kn + \frac{4}{3}^{k}k^2\right)$ (Balasubramanian et al. 1998)

Related work in parameterized complexity

- Zhou 2001 — treewidth/pathwidth and graph drawing
- Dujmović et al. 2001 — layered drawings

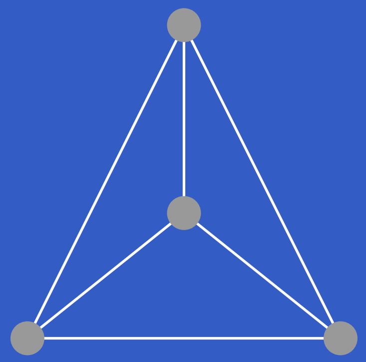
Modified goal

Develop a parameterized algorithm for upward planarity testing.

our parameter: the number of triconnected components.

k-connectivity

Definition. A graph is *k-connected* if there are at least k vertex-disjoint paths between any two vertices.



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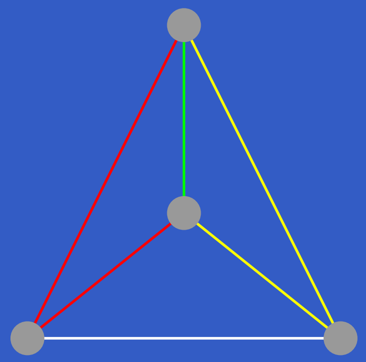

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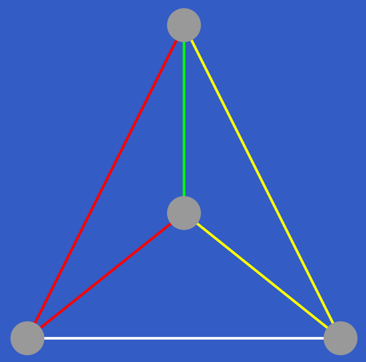
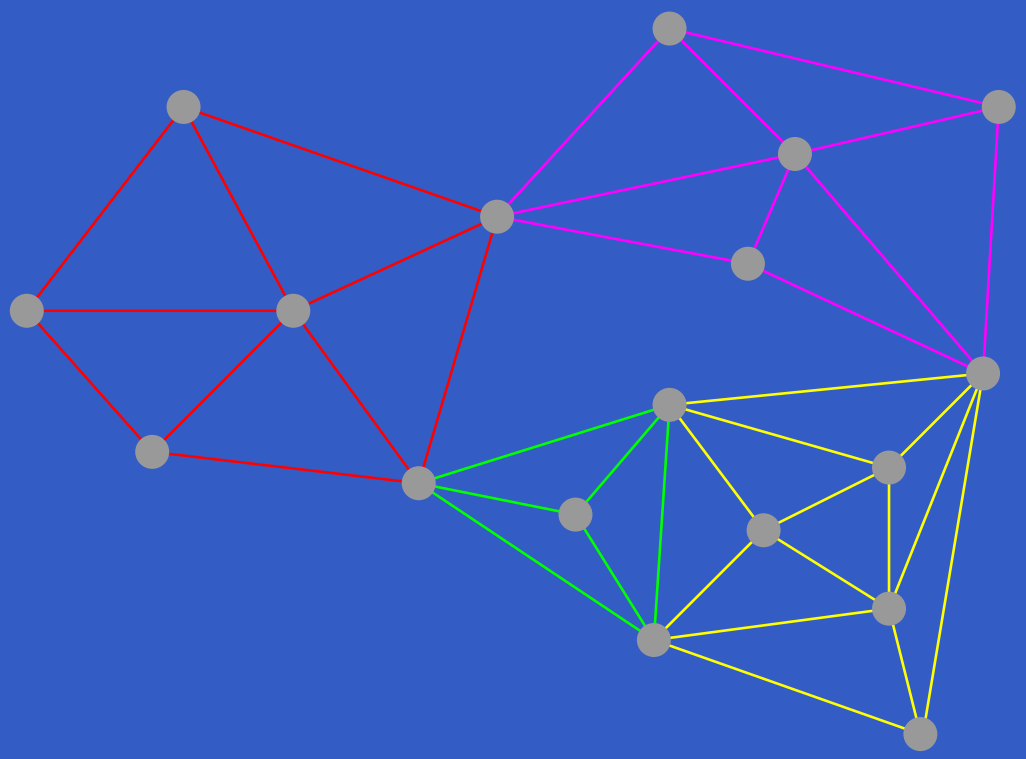


2-connected = biconnected
3-connected = triconnected

k-connected components

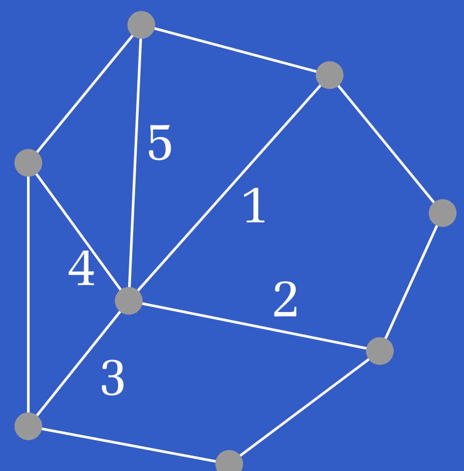
Definition. A k-connected component is a maximal k-connected subgraph

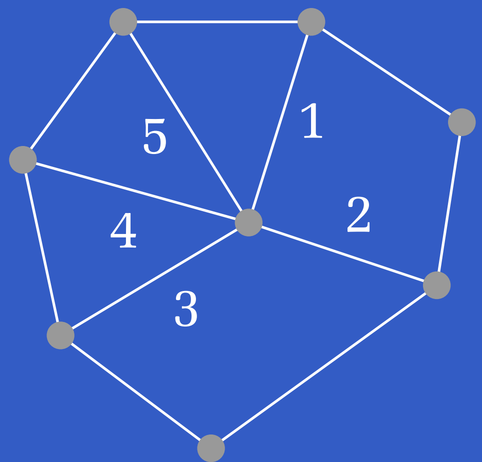


Preliminary definitions — Embeddings

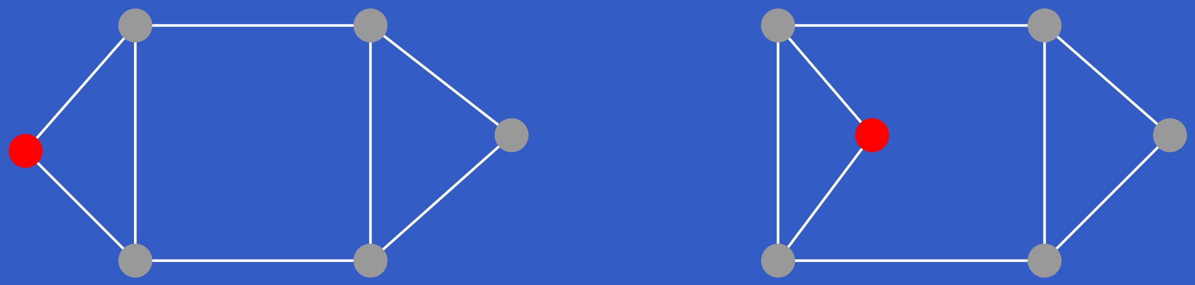
- Two different planar drawings may have similar structure
- An *embedding* is a description of this structure

Definition. The *(planar) embedding* associated with a drawing is the collection of clockwise orderings of the edges around each vertex.



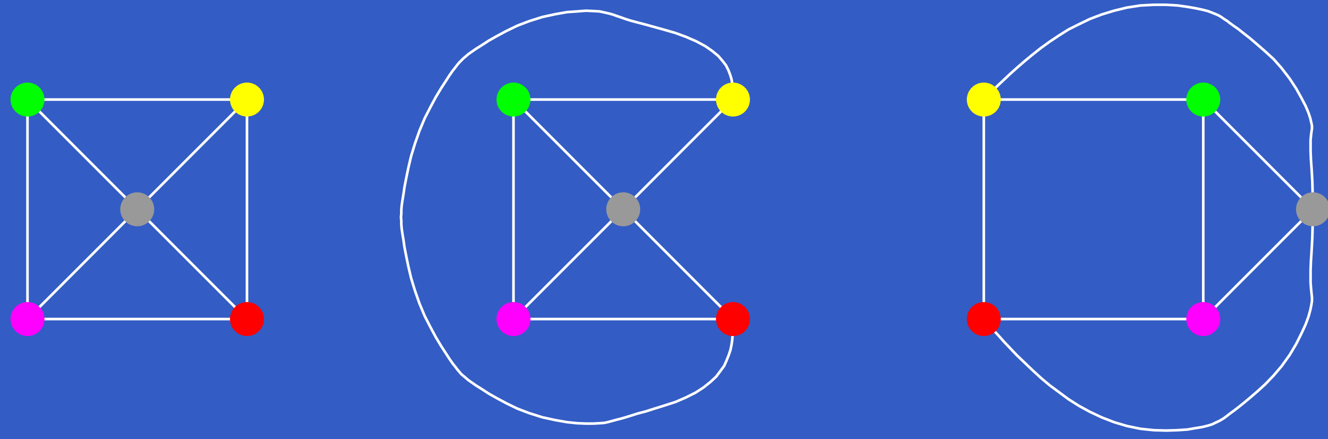


Embeddings



Equivalence of drawings

Definition. Two drawings are *equivalent* if they have the same embedding, and are *strongly equivalent* if the have the same embedding and the same outer face.

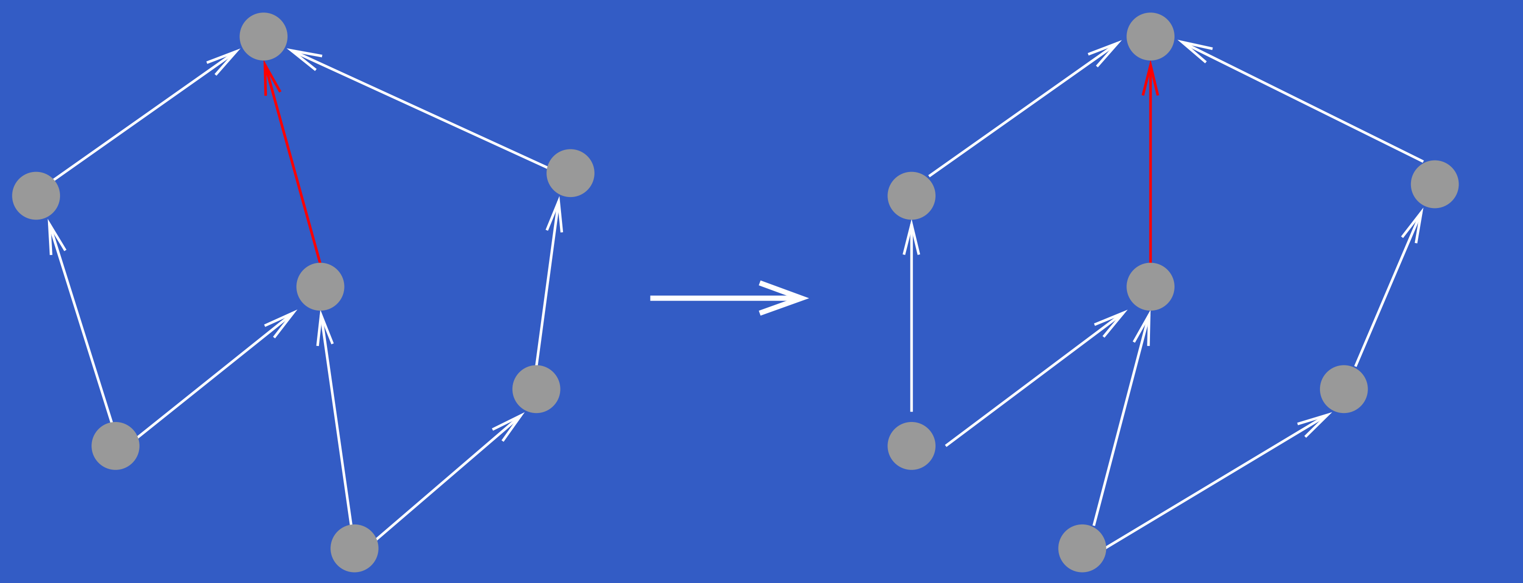


Outline

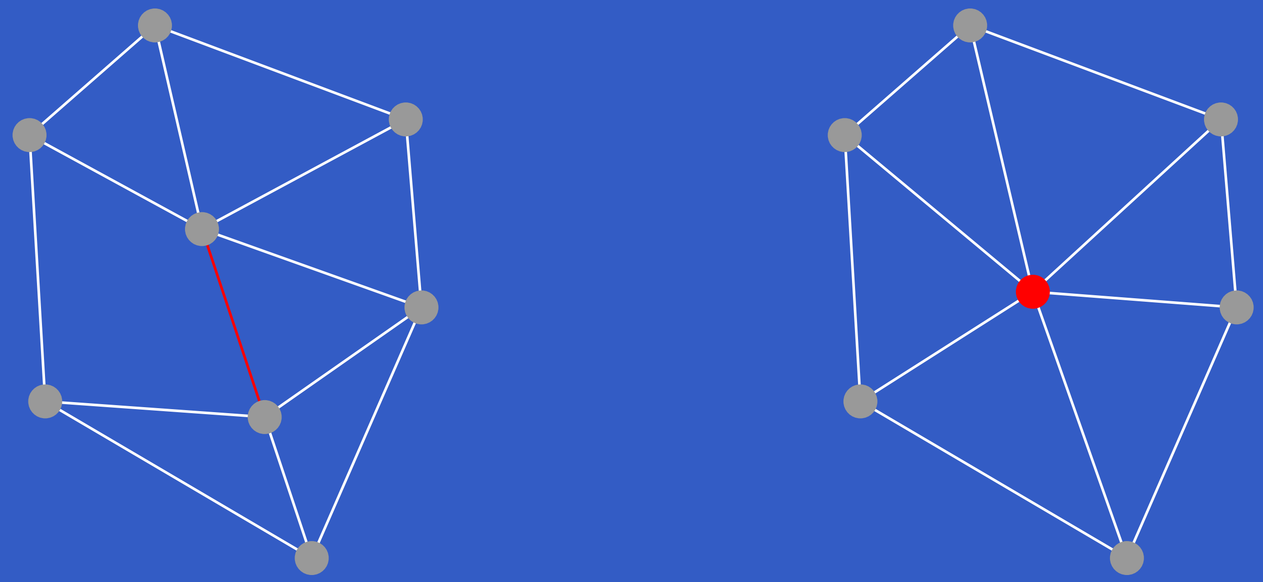
- Transformations of drawings
- Edge contraction
- Joining subgraphs
- Parameterized algorithm for biconnected graphs
- Conclusion

Transformations of drawings

- If a graph is upward planar, we can draw it so that a specified edge is drawn vertically
- We can scale and translate drawings, preserving upward planarity



Edge contraction

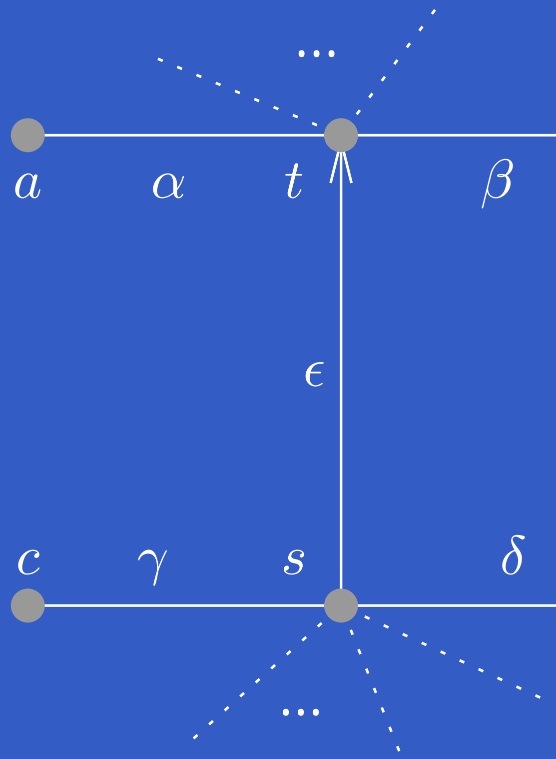


Edge contraction

Question: after contracting an edge ϵ, is the resulting embedding still upward planar?

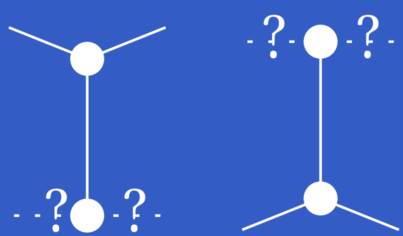
Edge contraction

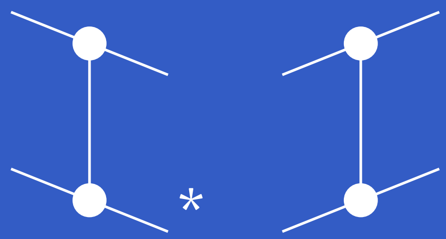
- look at the edge ordering neighbours

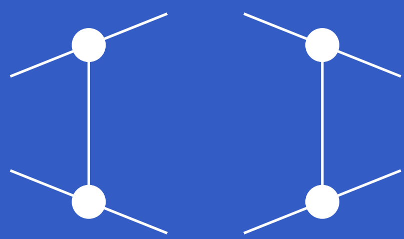


- consider all possibilities for the orientations of the neighbours

Is the contracted graph upward planar?

- yes:  (Hutton and Lubiw 1996)

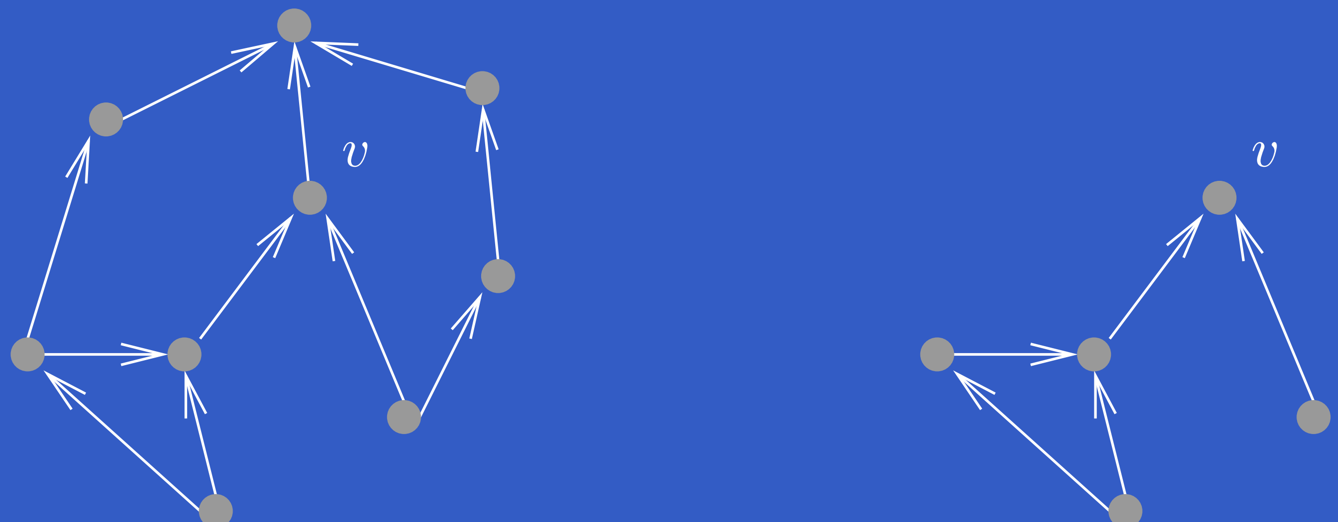


- no:  (corollary of Tamassia and Tollis 1986)
- if and only if $G_{\overleftarrow{\epsilon}}$ is upward planar: remaining cases

Edge contraction

Use characterization by Hutton and Lubiw:

Theorem. *Given ϕ, a planar drawing of a directed acyclic graph G, there is an upward planar drawing strongly equivalent to ϕ if and only if every vertex v is a sink on the outer face of ϕ_v.*

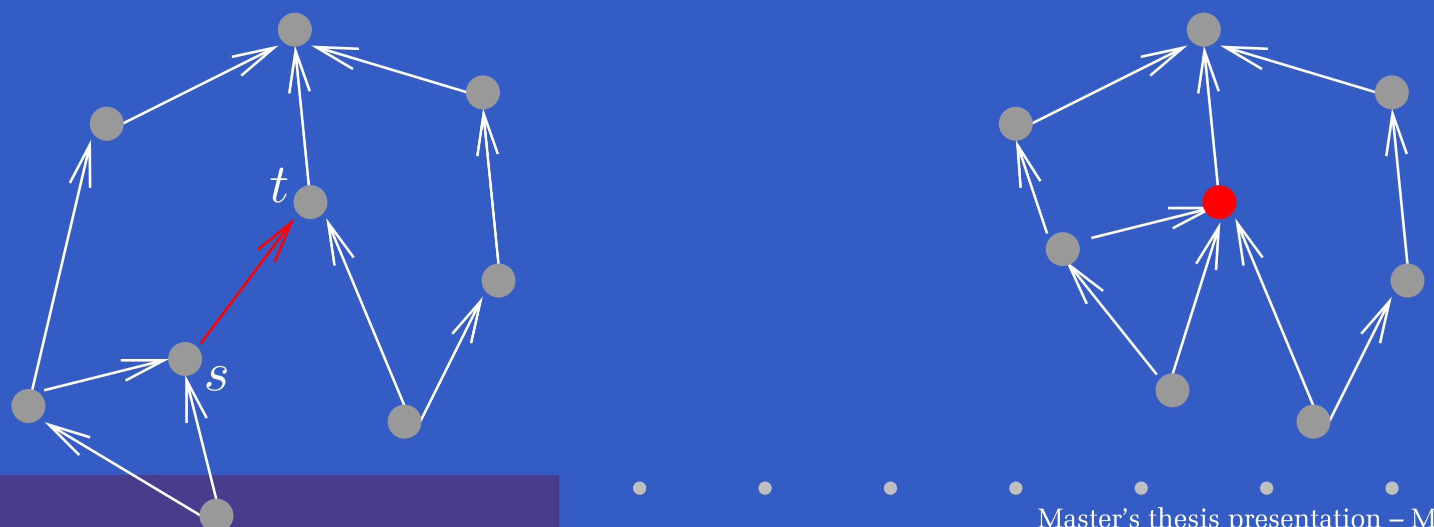


Proof outline

In the contracted graph:

- v is a sink (← only show this)
- G/ϵ is acyclic
- v is on the outer face

only need to consider vertices that have s as a predecessor but not t



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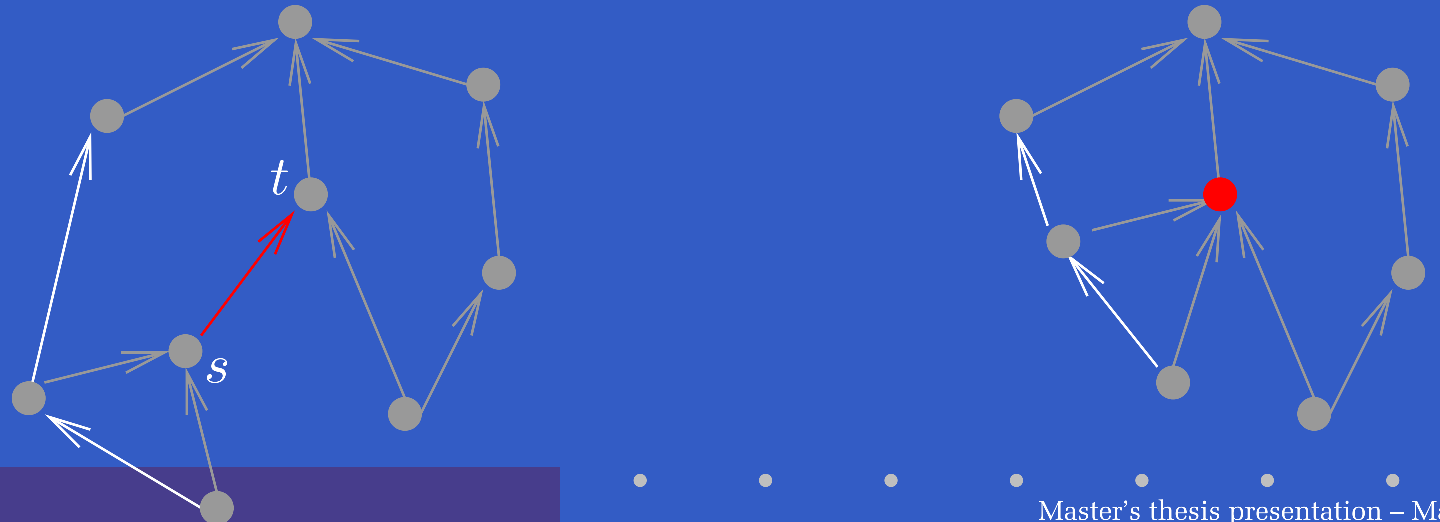

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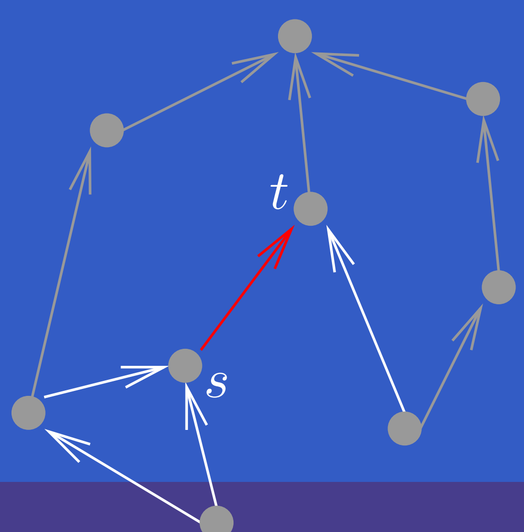

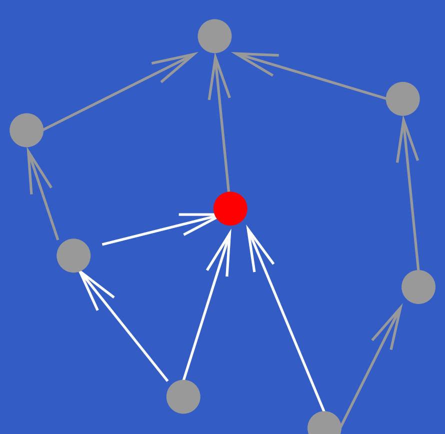

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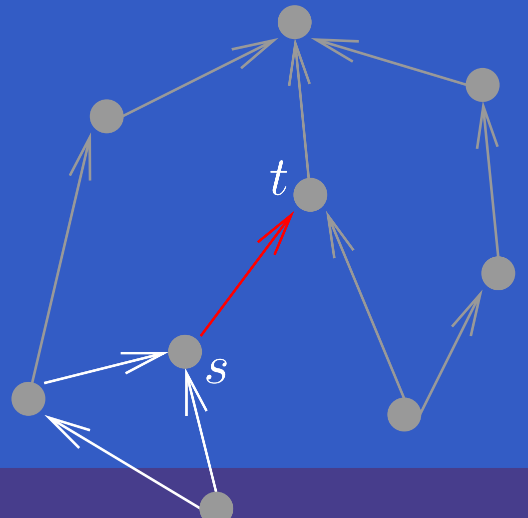
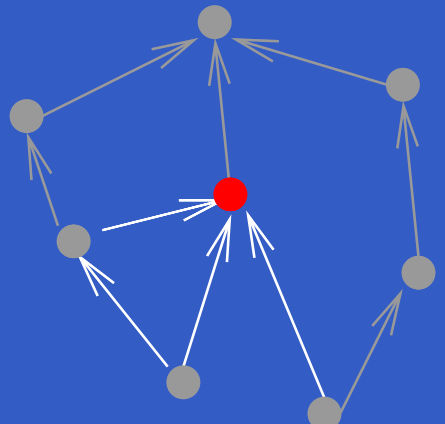
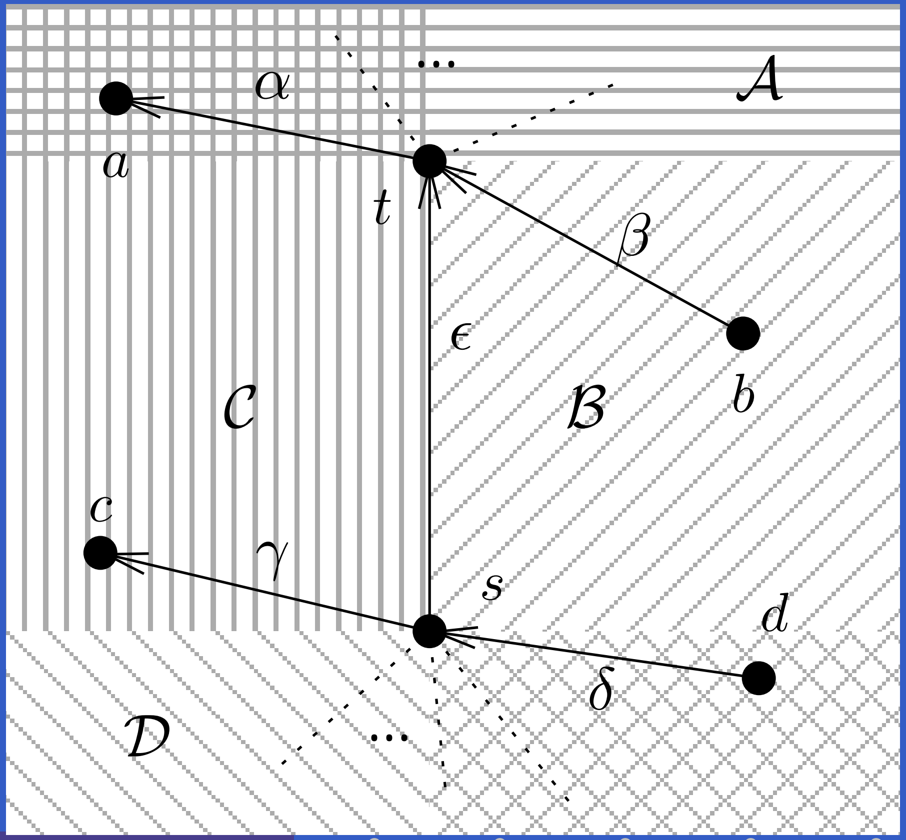




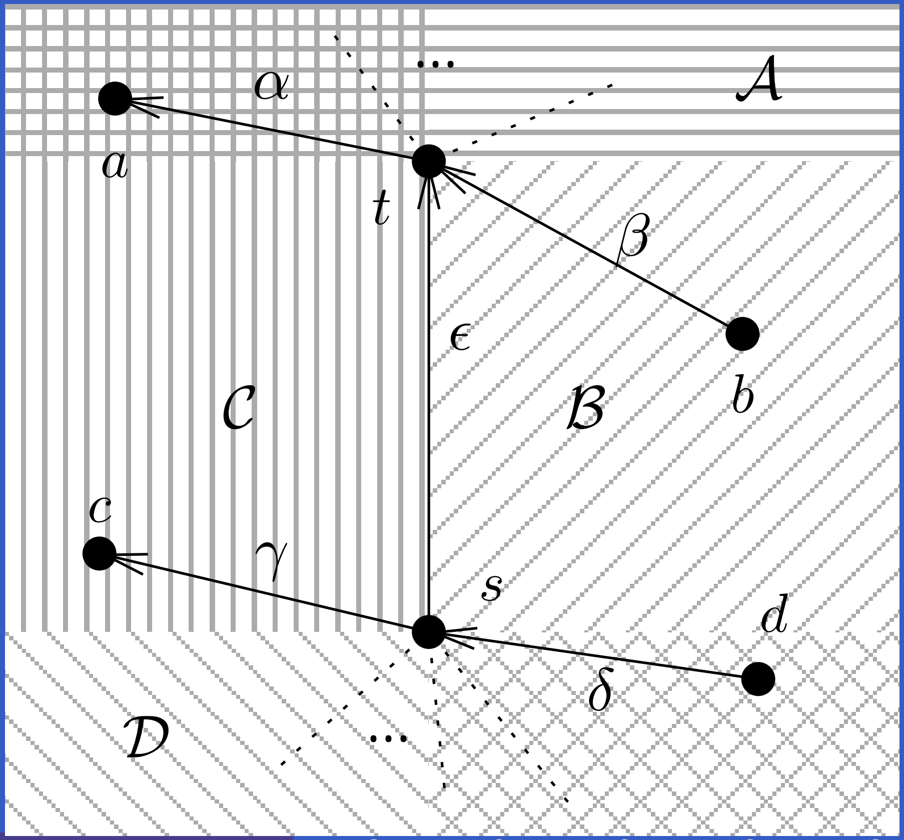
v is a sink

- we can draw ϵ vertically



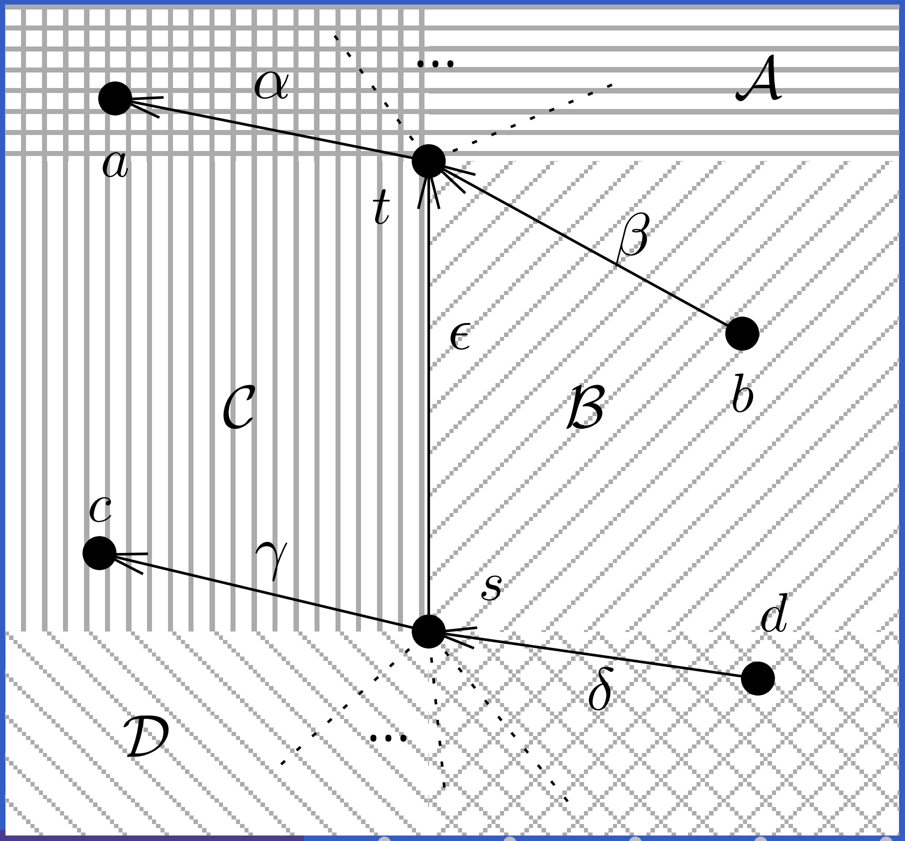
v is a sink

- we can draw ϵ vertically
- where can vertices be in relation to ϵ?



Locations of vertices

- predecessors of t must be in $\mathcal{B}$ or $\mathcal{D}$
- successors of s must be in $\mathcal{A}$ or $\mathcal{C}$

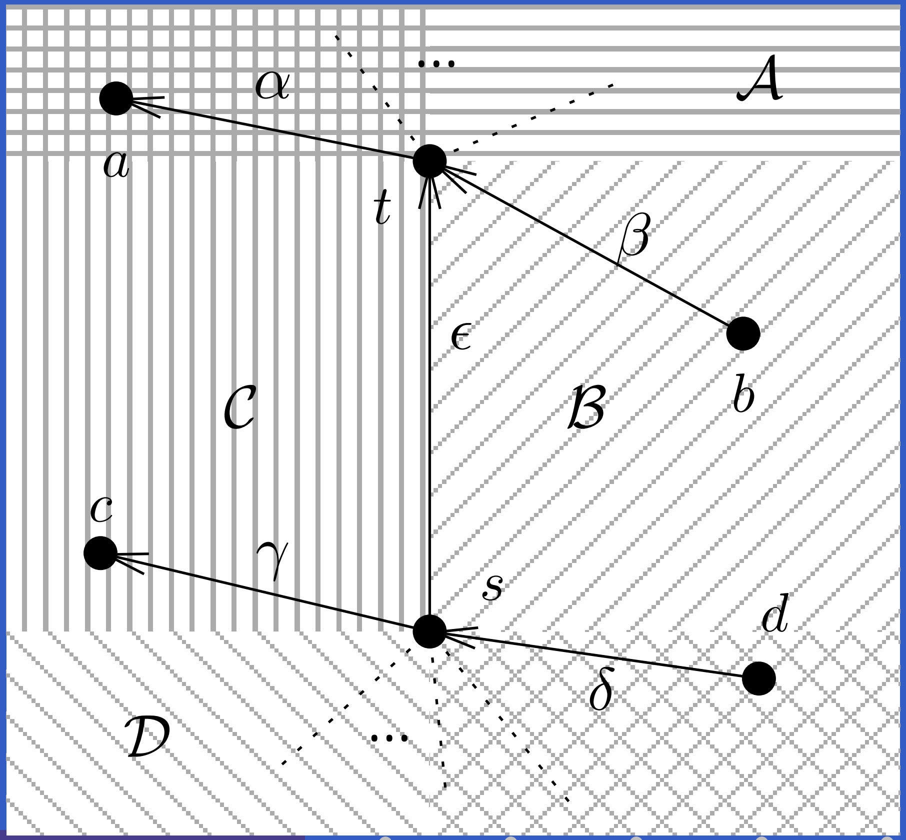


v is a sink

- if not: there is an outgoing edge (v, v_1)
- v was a sink in the original graph
- v_1 must be a predecessor of t
- v must be a predecessor of t

Where can v be drawn?

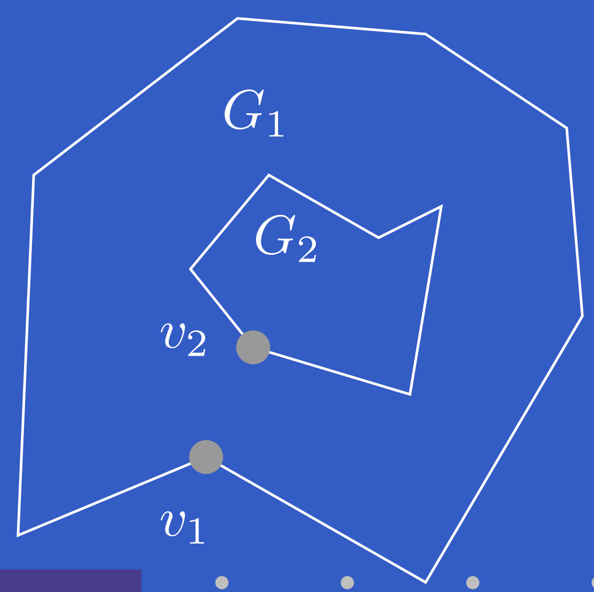
- v is a predecessor of t — must be in $\mathcal{B}$ or $\mathcal{D}$
- v is a successor of s — must be in $\mathcal{A}$ or $\mathcal{C}$



Joining subgraphs

Contracting edges allows us to join two upward planar subgraphs

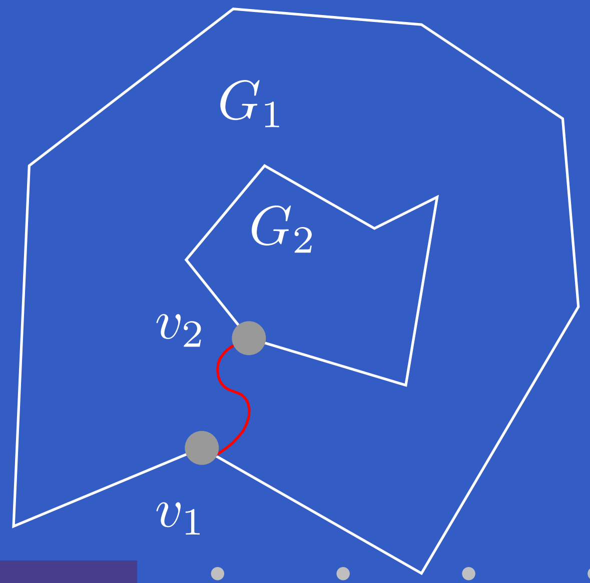
- draw G_1 and G_2



Joining subgraphs

Contracting edges allows us to join two upward planar subgraphs

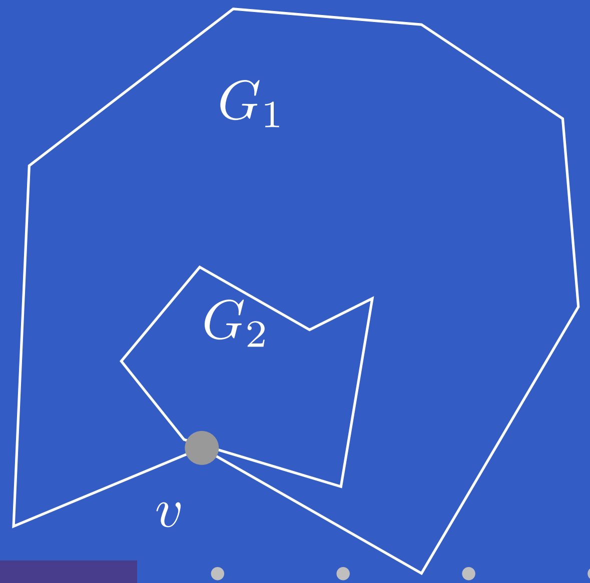
- draw G_1 and G_2
- draw a curve connecting v_1 and v_2



Joining subgraphs

Contracting edges allows us to join two upward planar subgraphs

- draw G_1 and G_2
- draw a curve connecting v_1 and v_2
- contract the edge (v_1, v_2)

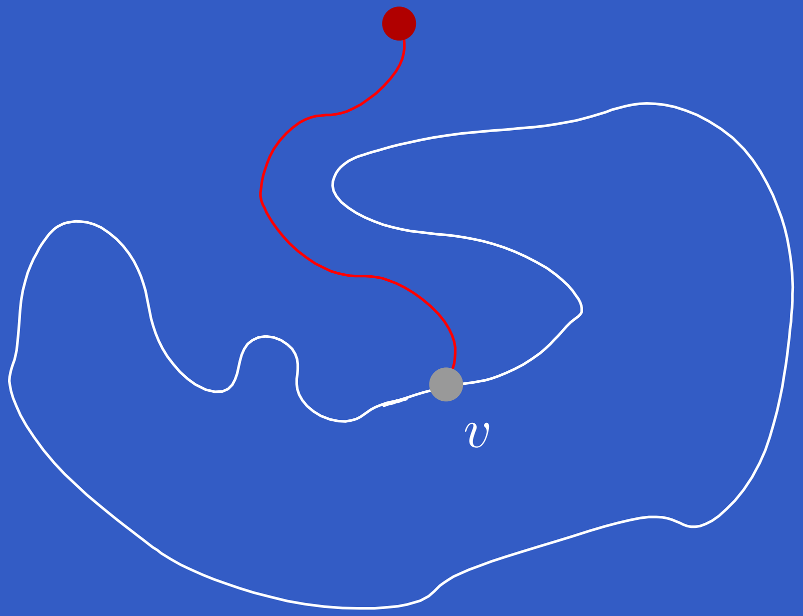


Joining subgraphs

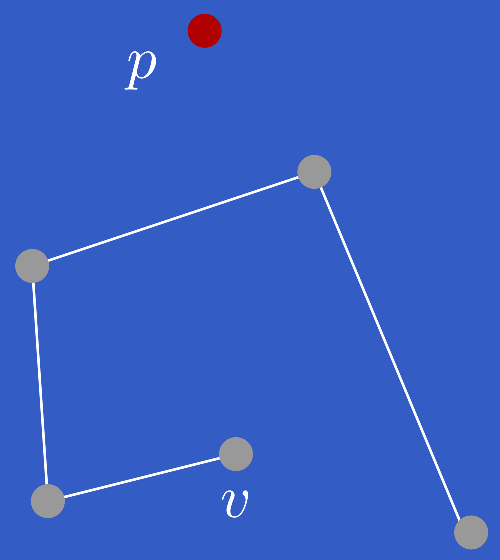
Goal: characterize when we can join two upward planar graphs to produce a new upward planar graph

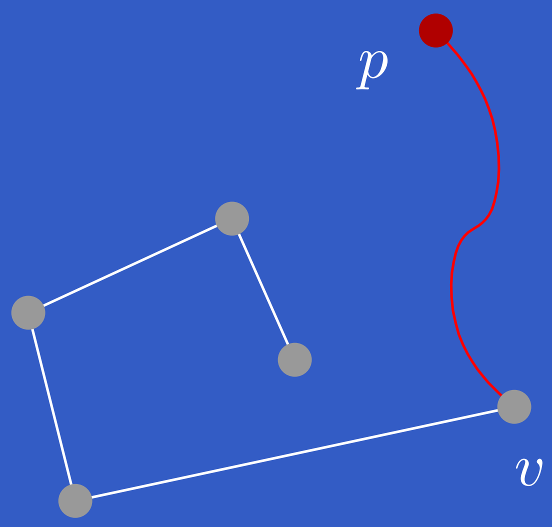
Visibility from above and below

v is *visible from above (below)* in an embedding Γ if there is a drawing corresponding to Γ in which a curve can be drawn from v to a point above (below) the drawing.



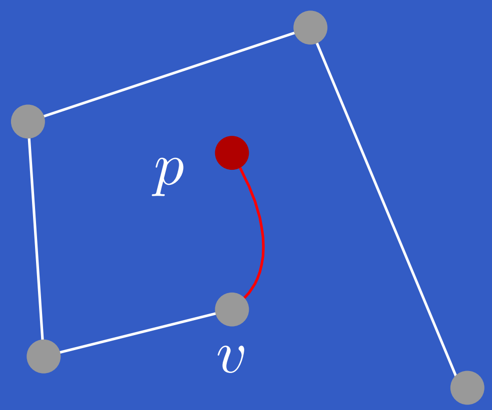
Visibility from above and below

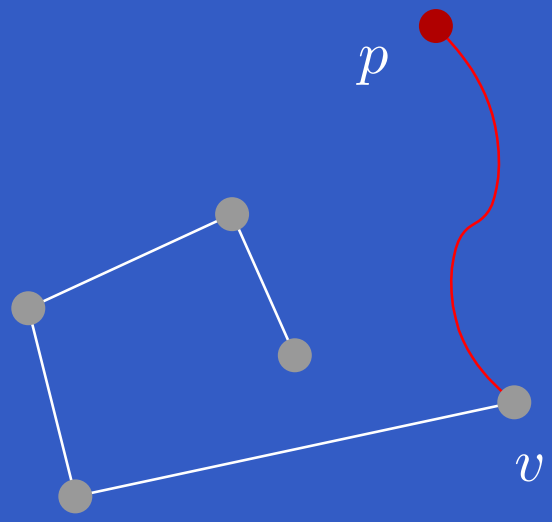




Alternate definition of visibility

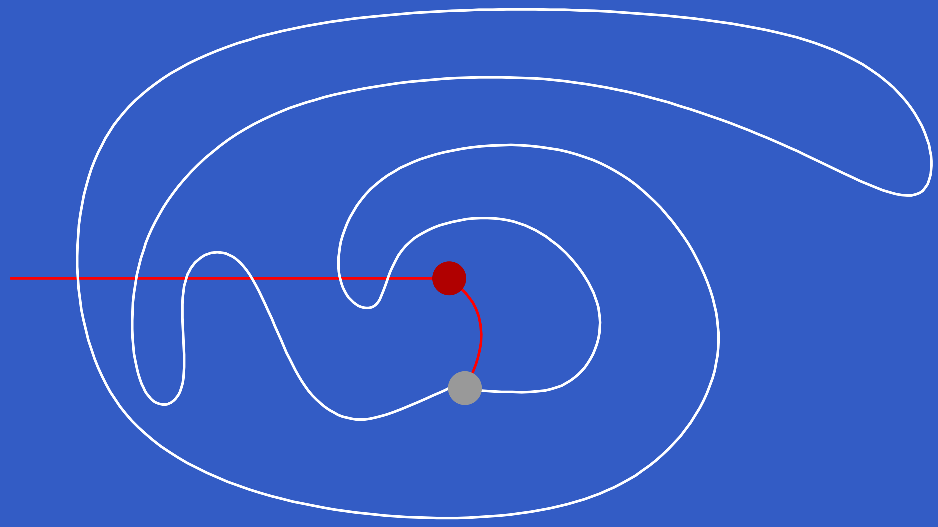
v is *visible from above* if a curve can be drawn from v to a point on the outer face above v





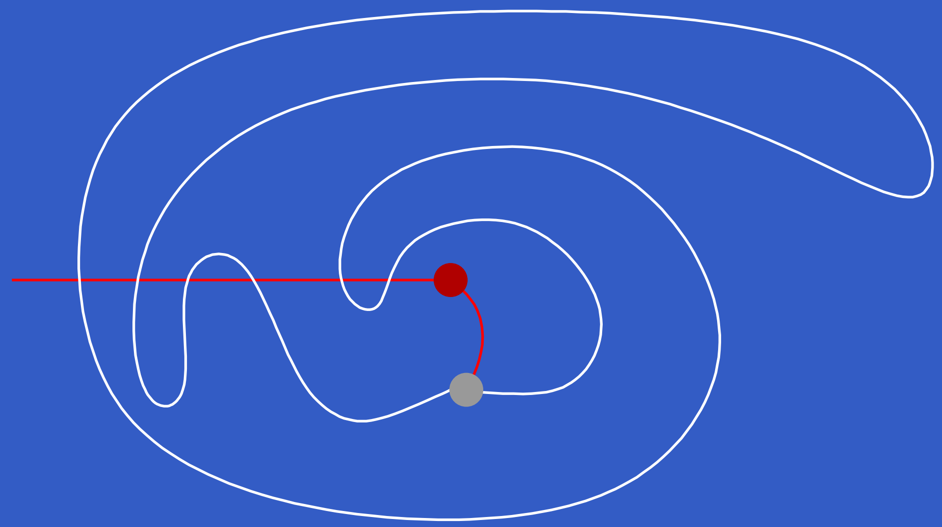
Transforming the drawing

- draw a horizontal ray ℓ from p
- count the number of times it crosses the boundary of the outer face



Removing crossings

- modify the drawing so that there are no crossings
- we define a procedure that reduces the number of crossings
- apply the procedure until no more crossings



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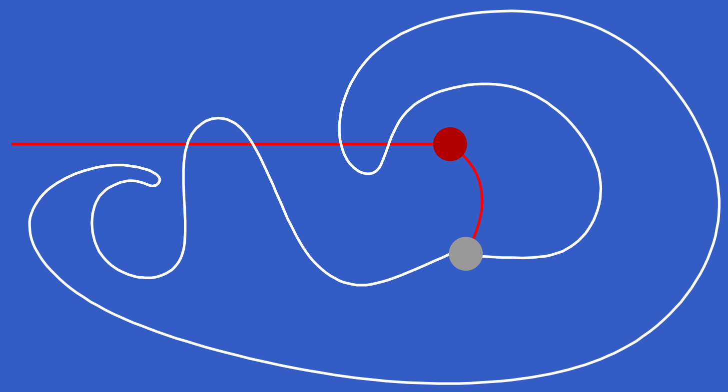

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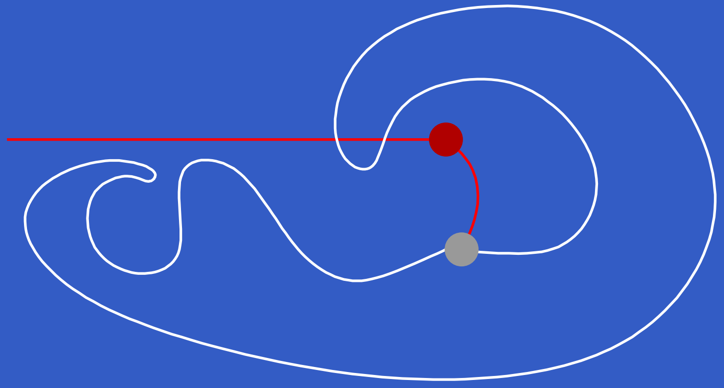

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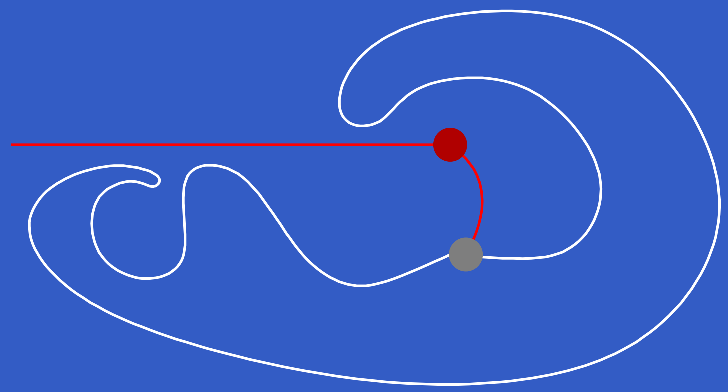


Visibility from above

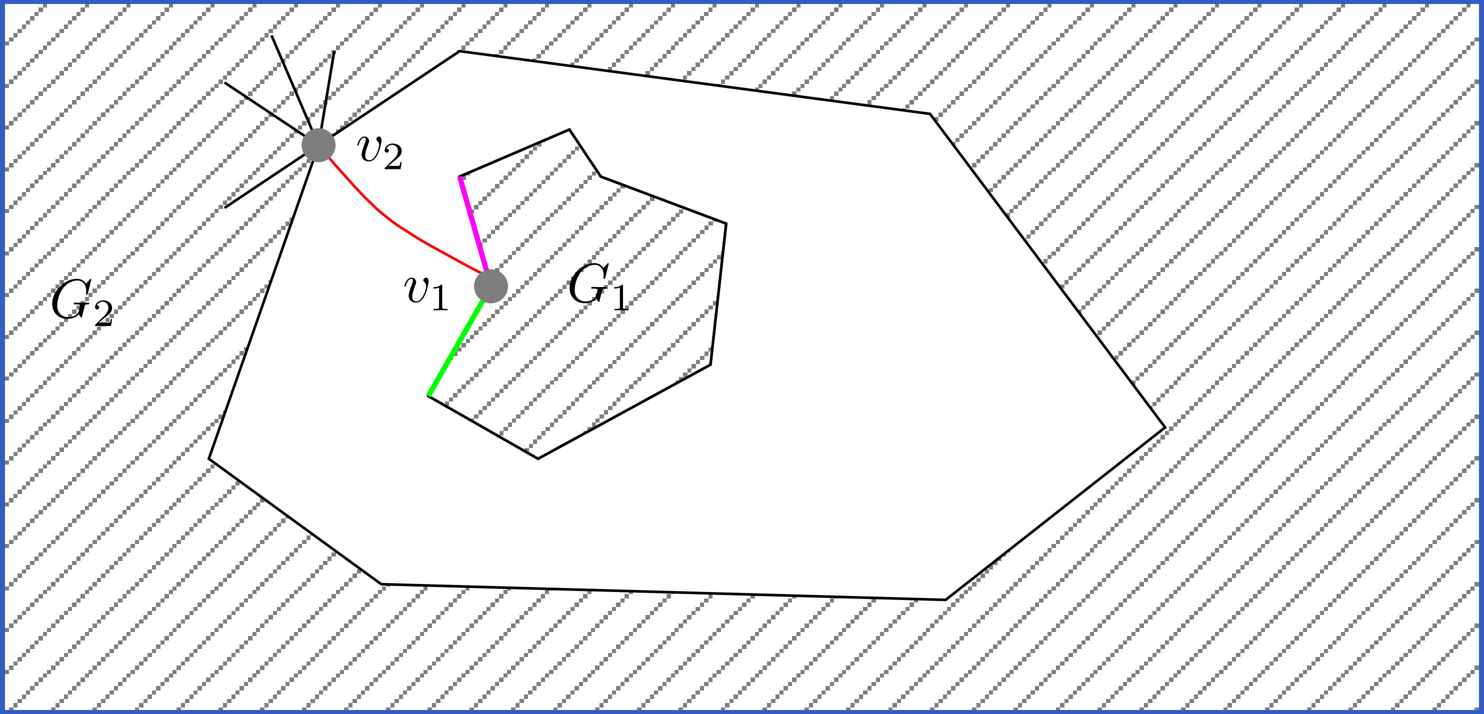
- We have shown that our two definitions of visibility are equivalent
- Our first definition allows us to join two graphs by drawing the edge (v_1, v_2)
- Our second definition allows us to determine the visibility of a vertex by looking at its incident edges
  - e.g. if v has an outgoing edge on the outer face, it is visible from above

Three cases for joining subgraphs

- v_1 and v_2 are both sources (or both sinks)
- v_1 is a source (or sink)
- neither is a source nor sink ($\leftarrow$ only show this)

Neither v_1 nor v_2 are sources nor sinks

If neither v_1 nor v_2 are cutvertices, G is upward planar if and only if v_1 or v_2 has an outgoing and an incoming edge on the outer face that are edge ordering neighbours



Joining subgraphs — conclusions

- We have characterizations for when we can join two upward planar graphs to obtain a larger upward planar graph.
- This allows us, in some cases, to join upward planar biconnected graphs.

Biconnected graphs

- a triconnected graph has a unique planar embedding (Diestel 2000)
- how many embeddings does a biconnected graph have?
- find a bound on the number of embeddings
- test each embedding for upward planarity (Bertolazzi et al. 1994)

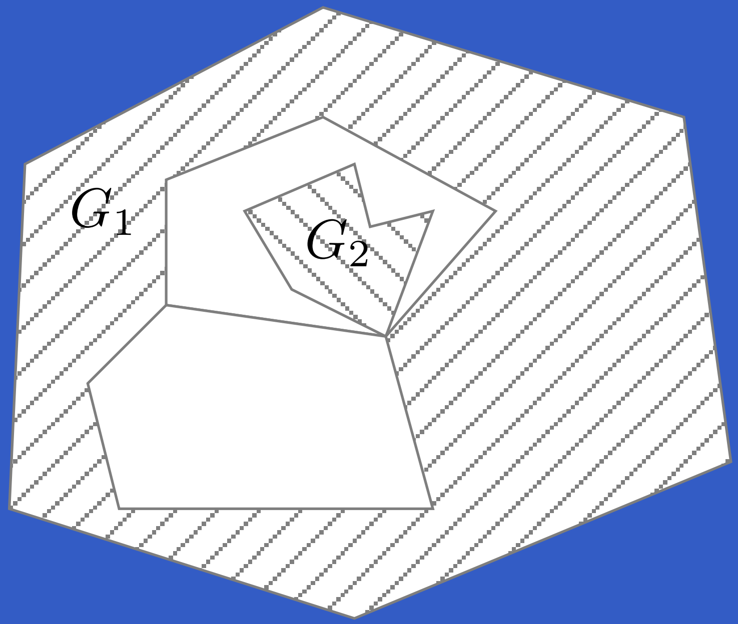
Biconnected graphs — outline

We obtain our bound by first considering a restricted case, and building up on this case.

- two triconnected components that share a common vertex
- k triconnected components that share a common vertex
- k triconnected components

Two triconnected components

Given two triconnected components that share at least one common vertex, how many possible embeddings do we have for the combined graph?



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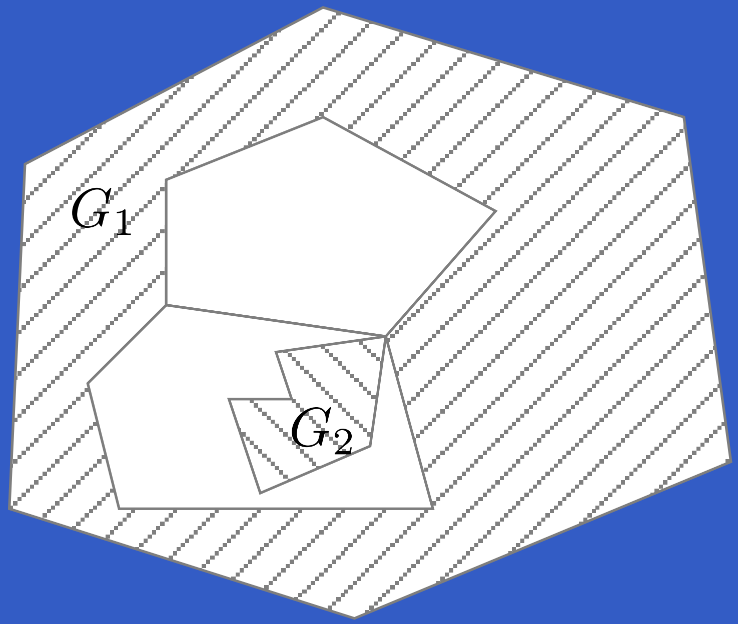

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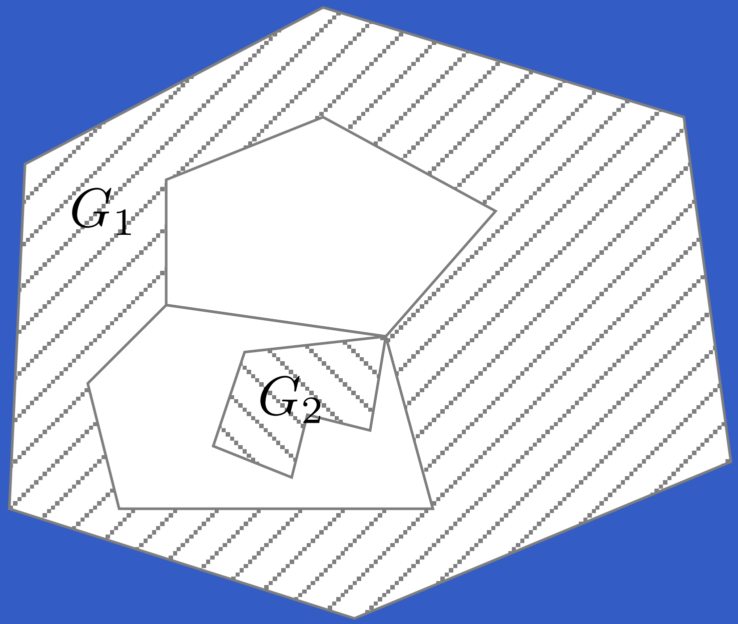


k triconnected components

Given k triconnected components that share at least one common vertex, how many possible embeddings do we have for the combined graph?

- similar to two triconnected components
- we must take into account the order of the components around the common vertex.

$(k-1)!8^{k-1}$ possibilities

k triconnected components

How many embeddings do we have for a biconnected graph with k triconnected components?

$k!8^{k-1}$

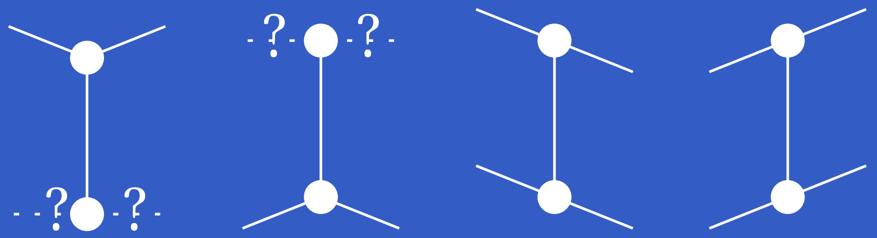
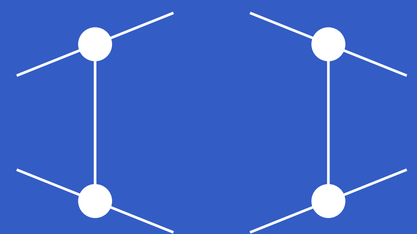
Algorithm

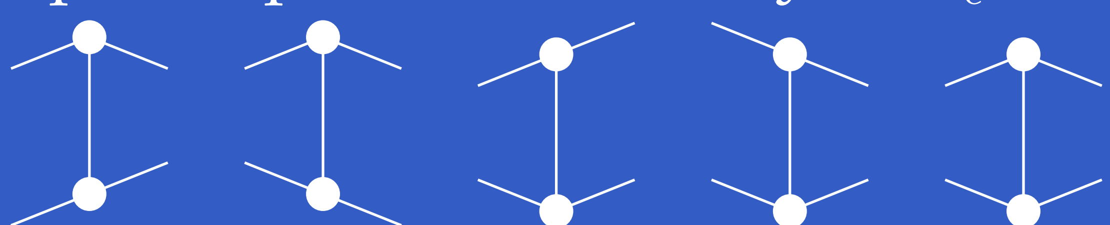
- split G into k triconnected components ($O(n^2)$ time — Hopcroft and Tarjan)
- for each possible embedding of G, and each possible outer face ($k!8^{k-1}n$ iterations)
    - test if the embedding is upward planar ($O(n^2)$ time — Bertolazzi et al.)

total time: $O(k!8^k n^3)$

Conclusions - edge contraction

The contracted graph is

- upward planar: 
- not upward planar: 
- upward planar if and only if $G_{\overleftarrow{\epsilon}}$ is upward planar:



Conclusions - joining subgraphs

Characterizations for

- v_1 and v_2 are both sources
- v_1 is a source
- neither is a source nor sink

Conclusions - biconnected graphs

- parameterized algorithm where the parameter k is the number of triconnected components.
- running time: $O(k!8^k n^3)$

Future work

- parameterized algorithm for general graphs
- explore other parameters, e.g. the number of sources and sinks
- upward planarity testing as a maximization problem
- more applications of parameterized complexity techniques to graph drawing problems